

Soliton solutions of the classical lattice sine-Gordon system

B. Enriquez

Abstract. We study the soliton-type solutions of the system introduced by B. Feigin and the author in [EF]. We show that it reduces to a top-like system, and we study the behaviour of the solutions at the lattice infinity. We compute the scattering of the solitons and study some periodic solutions of the system.

Introduction.

In this work, we study solutions to the lattice sine-Gordon system introduced by B.L. Feigin and the author in [EF]. This system consists of two families of compatible flows, analogues of the sine-Gordon and mKdV flows. We recall that the solutions studied in the continuous context are characterized by the requirement that they be stationary with respect to a linear combination of those flows. We study the analogous problem here, and reduce it to a top-like system. We then show how to pass from the group variables satisfying this system to the local variables (inverse scattering) ; in particular, we find solutions with variables tending to a constant at lattice infinity (soliton solutions). We then study the scattering of these solutions, and we find a purely elastic behaviour. Two features are rather different from the continuous case : the absence of charge of the solitons and their non-fermionic character (solutions with multiple poles exist). This is due to the difference between the relevant subalgebras of \widehat{sl}_2 : positive Cartan and negative principal subalgebras instead of positive and negative principal subalgebras in the continuous case.

Finally, we study some periodic solutions of the system. We find that such solutions are characterized by the condition that the difference of the two points at infinity on the spectral curve is torsion in its Jacobian.

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1. Review of continuous sine-Gordon system.

Let ϕ be a function on \mathbf{R} and let us consider the matrix $M_x(\lambda) = P \exp \int_{-\infty}^x (\Lambda + \phi h) \cdot \bar{n}_x$, with $\bar{n}_x = n_x \exp \sum_{\substack{i>0 \\ \text{odd}}} \int_x^{+\infty} u_i \Lambda^{-i}$, and $\partial_x + \Lambda + \phi h = n_x (\partial + \Lambda + \sum_{\substack{i>0 \\ \text{odd}}} u_i \Lambda^{-i}) n_x^{-1}$,
 $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\Lambda = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}$ ([DS]) ; $P \exp \int_{-\infty}^x (\Lambda + \phi h)$ is developed around $\lambda = 0$ and takes the form

$$\begin{pmatrix} 1 + \lambda \int_{-\infty}^x e^{2\varphi(y)} \int_{-\infty}^y e^{-2\varphi} + \dots & \int_{-\infty}^x e^{2\varphi} + \dots \\ \lambda \int_{-\infty}^x e^{-2\varphi} + \dots & 1 + \lambda \int_{-\infty}^x e^{-2\varphi(y)} \int_{-\infty}^y e^{2\varphi} + \dots \end{pmatrix} \begin{pmatrix} e^{\varphi(x)} & 0 \\ 0 & e^{-\varphi(x)} \end{pmatrix}.$$

Then the mKdV flows are $\partial_n^{KdV} M_x = M_x \Lambda^{2n+1}$, $n \geq 0$ and the sine-Gordon flow is $\partial_0^{SG} M_x = \Lambda^{-1} M_x$. We introduce higher sine-Gordon flows by $\partial_n^{SG} M_x = \Lambda^{-(2n+1)} M_x$. We have

$$\partial_x \partial_0^{SG} \varphi(x) = e^{2\varphi(x)} - e^{-2\varphi(x)},$$

$$\begin{aligned}\partial_x \partial_1^{SG} \varphi(x) &= e^{2\varphi(x)} \int_{-\infty}^x e^{-2\varphi(y)} \int_{-\infty}^y (e^{2\varphi} - e^{-2\varphi}) - e^{-2\varphi(x)} \int_{-\infty}^x e^{2\varphi(y)} \int_{-\infty}^y (e^{2\varphi} - e^{-2\varphi}), \\ \partial_x \partial_n^{SG} \varphi(x) &= e^{2\varphi(x)} \int_{-\infty}^x e^{-2\varphi(y)} \partial_y \partial_{n-1}^{SG} \varphi(y) - e^{-2\varphi(x)} \int_{-\infty}^x e^{2\varphi(y)} \partial_y \partial_{n-1}^{SG} \varphi(y).\end{aligned}$$

These equations are well defined for functions satisfying the boundary conditions $e^{4\varphi(x)} \xrightarrow[x \rightarrow -\infty]{} 1$ faster than any inverse polynomial, and they preserve these boundary conditions. If φ is such that in addition $e^{4\varphi(x)} \xrightarrow[x \rightarrow +\infty]{} 1$ faster than any inverse polynomial, and if the conditions

$$\int_{-\infty}^{\infty} (e^{2\varphi} - e^{-2\varphi}) = \int_{-\infty}^{\infty} (e^{2\varphi(x)} \int_{-\infty}^x e^{-2\varphi} - e^{-2\varphi(x)} \int_{-\infty}^x e^{2\varphi}) = \dots = 0$$

are met (they are a rewriting of $[\Lambda, P \exp \int_{-\infty}^{\infty} (\Lambda + \phi h)] = 0$), then the higher SG equations can be rewritten with $+\infty$ replacing $-\infty$, at point φ ; these conditions are preserved by the flows. [Note that any solution to this system, such that $e^{4\varphi} \rightarrow 1$ and $\partial_k^{SG} \varphi \rightarrow 0$, faster than any $\frac{1}{x^k}$ when $x \rightarrow \pm\infty$ will also satisfy the conditions $\int_{-\infty}^{\infty} (e^{2\varphi} - e^{-2\varphi}) = \dots = 0$, since we deduce from $\partial_k^{SG}(\varphi h) = (\text{Ad}^{-1}(P \exp \int_{-\infty}^x (\Lambda + \phi h))(\Lambda^{-(2k+1)}))_{h\lambda^0}$ that $\text{Ad}^{-1}(P \exp \int_{-\infty}^{\infty} (\Lambda + \phi h))(\Lambda^{-1})$ has diagonal elements equal to zero, and from $\partial_k^{SG}(1 + \phi \Lambda^{-1}) = \text{degree } -1$ part of $\text{Ad}^{-1}(P \exp \int_{-\infty}^x (\Lambda + \phi h))(\Lambda^{-(2k+1)})$ we deduce that $\text{Ad}^{-1}(P \exp \int_{-\infty}^x (\Lambda + \phi h))(\Lambda^{-1})$ is proportional to Λ^{-1} ; since its square equals Λ^{-2} , it is equal to Λ^{-1} . These two equations are consequences of $\partial_k^{SG} M_x = M_x(M_x^{-1} \Lambda^{-(2k+1)} M_x)_{\geq 0}$ and $\partial_k^{SG} n_x = (M_x^{-1} \Lambda^{-(2k+1)} M_x)_{< 0} n_x$. The first example of this statement is $\int_{-\infty}^{\infty} (e^{2\varphi} - e^{-2\varphi}) = 0$; it is a consequence of $\int_{-\infty}^{\infty} \partial_x \partial_0^{SG} \varphi = \partial_0^{SG} [\varphi(\infty) - \varphi(-\infty)] = 0$ (conservation of topological charge).]

The higher sine-Gordon flows commute with each other and with the mKdV flows. The solution of this system is formally written

$$M_x(\lambda; t_n^{SG}, t_m^{KdV}) = \exp\left(\sum_{n \geq 0} t_n^{SG} \Lambda^{-(2n+1)}\right) g_0 \exp\left(\sum_{m \geq 0} t_m^{KdV} \Lambda^{2m+1}\right).$$

Then to extract e^φ from this matrix, we have to take matrix coefficients of this group element in some integrable representation. It means that $M_x(\lambda)$ has to be replaced by a variable $\widetilde{M}_x(\lambda)$ living in the central extension of the loop group, obeying the same equations as $M_x(\lambda)$. Now the Cartan part of $\widetilde{M}_x(\lambda)$ is $\exp(\varphi h)$ times a central term: it will be equal to $\langle \Lambda_0 | \widetilde{M}_x(t_n^{SG}, t_m^{KdV}) | \Lambda_0 \rangle$ if $|\Lambda_0\rangle$ is a level-one highest weight vector, such that $h|\Lambda_0\rangle = 0$. Now, if $|\Lambda_1\rangle$ is a level-one highest weight vector, with $h|\Lambda_1\rangle = |\Lambda_1\rangle$, we have

$$e^{\varphi(t_n^{SG}, t_m^{KdV})} = \frac{\langle \Lambda_1 | \widetilde{M}_x(t_n^{SG}, t_m^{KdV}) | \Lambda_1 \rangle}{\langle \Lambda_0 | \widetilde{M}_x(t_n^{SG}, t_m^{KdV}) | \Lambda_0 \rangle}.$$

(cf. [OTU]).

In the case of the soliton solutions, we take $g_0 = e^{Q_1 F(z_1)} \dots e^{Q_n F(z_n)}$, with $[\Lambda^n, F(z)] = z^n F(z)$. Let $\tau_i(t_n^{SG}, t_m^{KdV}) = \langle \Lambda_1 | \widetilde{M}_x(t_n^{SG}, t_m^{KdV}) | \Lambda_1 \rangle$, then $\partial_n^{SG} \tau_i = t_n^{SG} \tau_i + \sum_{i=1}^n z_i^{-n} \frac{\partial \tau_i}{\partial Q_i}$, because $[\Lambda^{-n}, g_0] = \sum_{i=1}^n z_i^{-n} \frac{\partial g_0}{\partial Q_i}$, so, $\partial_n^{SG}(e^\varphi) = \sum_{i=1}^n z_i^{-n} \frac{\partial}{\partial Q_i}(e^\varphi)$; similarly, $\partial_n(e^\varphi) = \sum_{i=1}^n z_i^n \frac{\partial}{\partial Q_i}(e^\varphi)$. This shows :

Proposition.— *The mKdV and higher SG flows span a finite dimensional family of vector fields on the multisoliton solutions.*

Note also that g_0 can be considered as a monodromy matrix ; from its expression $\prod_{i=1}^n \exp(\sum_{k \in \mathbf{Z}} h \Lambda^k z_i^{-k})$ we deduce that as a function of λ it has poles at $\lambda = z_i$.

2. The lattice sine-Gordon system.

Let as in [EF] $x_i, y_i (i \in \mathbf{Z})$ be a system of lattice variables, with Poisson brackets $\{x_i, x_j\} = x_i x_j, \{y_i, y_j\} = y_i y_j, i < j, \{x_i, y_j\} = -x_i y_j, i \leq j$. Consider the matrix

$$g = \prod_{i=-\infty}^0 \begin{pmatrix} 1 & 0 \\ \lambda x_i & 1 \end{pmatrix} \begin{pmatrix} 1 & y_i \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1/\lambda x_0 + 1/y_1 \dots & 1 \end{pmatrix} \begin{pmatrix} 1 & [y_{-1} + \dots]^{-1} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & (\prod_{i \geq 1} \frac{\lambda x_i}{\lambda x_i + 1/y_i \dots} \frac{y_i}{y_i + 1/\lambda x_i + \dots})^{-1} \\ 0 & (\prod_{i \geq 1} \frac{\lambda x_i}{\lambda x_i + 1/y_i \dots} \frac{y_i}{y_i + 1/\lambda x_i + \dots})^{-1} \end{pmatrix}.$$

The lattice sine-Gordon and mKdV flows can be expressed as

$$(1) \quad \partial_n^{SG} g = \Lambda^{-(2n+1)} g \quad \text{and} \quad \partial_n^{KdV} g = gh\lambda^n, n \geq 0 ;$$

here $\Lambda = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}$. Let M be the matrix $\prod_{i=-\infty}^0 \begin{pmatrix} 1 & 0 \\ \lambda x_i & 1 \end{pmatrix} \begin{pmatrix} 1 & y_i \\ 0 & 1 \end{pmatrix}$.

Proposition.— *The following relation is satisfied*

$$\{M \otimes, g\} = \rho(M \otimes g) ,$$

where $\rho = -2 \sum_{i \geq 0} e_i \otimes f_{-i} - 2 \sum_{i \geq 1} f_i \otimes e_{-i} - \sum_{i \geq 1} h_i \otimes h_{-i}$. [Here $e_i = \begin{pmatrix} 0 & \lambda^i \\ 0 & 0 \end{pmatrix}, f_i = \begin{pmatrix} 0 & 0 \\ \lambda^i & 0 \end{pmatrix}, h_i = \begin{pmatrix} \lambda^i & 0 \\ 0 & -\lambda^i \end{pmatrix}]$

Proof. For the matrix elements of M corresponding to the simple roots ($\sum_{-\infty}^\infty x_i$ and $\sum_{-\infty}^\infty y_i$), this statement is $\{\sum_{-\infty}^\infty x_i, g\} = -2e_{-1}g, \{\sum_{-\infty}^\infty y_i, g\} = -2f_0g$, and it is a consequence of $\{\sum_{-\infty}^\infty x_i, g\}' = -2(ge_{-1}g^{-1})_-g, \{\sum_{-\infty}^\infty y_i, g\}' = -2(gf_0g^{-1})_-g$ which follow from [EF] (here $\{a, b\}' = \{a, b\} - (\deg a)(\deg b)ab, \deg x_i = 1, \deg y_i = -1$). We then check its compatibility with $\{M \otimes, M\} = r^L(M \otimes M) - (M \otimes M)r^R$: it is $(\delta \otimes 1)(\rho) = [\rho^{23}, \rho^{13}]$, (where $\delta(x) = [r, x \otimes 1 + 1 \otimes x]$). Recall that $\rho \in \hat{b}_+ \otimes \hat{b}_-$. Let $\xi, \eta \in \hat{b}_-$, this equality means that $\langle \rho, [\xi, \eta] \otimes 1 \rangle =$

$[\langle \rho^{13}, \xi^1 \rangle, \langle \rho^{23}, \eta^2 \rangle]$; since $[\rho, \xi \otimes 1] = \xi$ this equality is true (we have used the duality between \hat{b}_+ and \hat{b}_-). ■

Let us set $H_i = \text{constant coefficient of } \text{tr}(\Lambda^{-(2i+1)} M)$. The family H_i is Poisson commutative. Then

$$\{H_i, g\} = \sum_j -2d_{i-j}(f_{-j}^L g) - 2a_{i-j}(e_{-j-1}^L g) - (c_{i-j} - b_{i+1-j})(h_{-j}^L g),$$

where we have set $M = \begin{pmatrix} a(\lambda) & c(\lambda) \\ b(\lambda) & d(\lambda) \end{pmatrix}$ and $a(\lambda) = \sum_{i \geq 0} a_i \lambda^i$, etc...

The Hamiltonian flows generated by the H_i commute with the mKdV flows (since H_i are in the Poisson algebra generated by b_1 and c_0 , which commute to the integrals of motions); these flows will be denoted ∂_{H_i} . Generally they will make sense only at the level of formal variables (whereas $\partial_n^{SG} = f_{-n}^L + e_{-n-1}^L$ will have concrete realizations), because a_{i-j} and d_{i-j} will be infinite. Nevertheless, for solutions of the system such that $x_n, y_n \xrightarrow{\pm\infty} a$, we can consider as we did in the continuous case that $a_i - d_i = b_i - c_{i-1} = 0$ and that the linear spans of $\partial_{H_1}, \dots, \partial_{H_n}$ and $\partial_1^{SG}, \dots, \partial_n^{SG}$ are the same.

Let us now show how the vector fields ∂_n^{SG} operate on the space of lattice variables, rapidly tending to a at infinity. Let us express those fields as

$$\partial_1^{SG}(\ln x_n) = -2 \sum_{i < n} y_i + 2 \sum_{i < n} x_i + x_n = (x_n - y_{n-1}) - (y_{n-1} - x_{n-1}) + (x_{n-1} - y_{n-2}) \dots,$$

$$\partial_1^{SG}(\ln y_n) = (y_n - x_n) - (x_n - y_{n-1}) + \dots,$$

and $\partial_n^{SG}(\ln x_n)$, $\partial_n^{SG}(\ln y_n)$ are given by (1), where the matrices

$$\text{Ad}^{-1} \prod_{j=-\infty}^i \begin{pmatrix} 1 & 0 \\ -\lambda x_j & 1 \end{pmatrix} \begin{pmatrix} 1 & -y_j \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & \lambda^{-1} \\ 1 & 0 \end{pmatrix},$$

$$\text{Ad}^{-1} \prod_{j=-\infty}^i \begin{pmatrix} 1 & -y_j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\lambda x_{j+1} & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & \lambda^{-1} \\ 1 & 0 \end{pmatrix}$$

are understood as

$$(3) \quad \lim_{N \rightarrow +\infty} \text{Ad}^{-1} \prod_{j=-N}^i \begin{pmatrix} 1 & 0 \\ -\lambda x_j & 1 \end{pmatrix} \begin{pmatrix} 1 & -y_j \\ 0 & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{1 + \frac{a^2 \lambda}{4}}} \begin{pmatrix} \frac{a}{2} & \lambda^{-1} \\ 1 & -\frac{a}{2} \end{pmatrix}$$

and

$$(4) \quad \lim_{N \rightarrow +\infty} \text{Ad}^{-1} \begin{pmatrix} 1 & 0 \\ -\lambda x_{-N} & 1 \end{pmatrix} \prod_{j=-N}^i \begin{pmatrix} 1 & -y_j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\lambda x_{j+1} & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{1 + \frac{a^2 \lambda}{4}}} \begin{pmatrix} \frac{a}{2} & \lambda^{-1} \\ 1 & -\frac{a}{2} \end{pmatrix}$$

(these limits are well defined, because of the assumptions on x_i, y_i). Then $\partial_{k+1}^{SG}(\ln x_n^{(\varepsilon)})$ is a linear combination of $\sum_{2n+\varepsilon \geq 2i_1+\varepsilon_1+k_1 \geq 2i_2+\varepsilon_2+k_2 \geq \dots} x_{i_1}^{(\varepsilon_1)} \cdots x_{i_{2k}}^{(\varepsilon_{2k})} \partial_1^{SG} \ln x_{i_{2k+1}}^{(\varepsilon_{2k+1})}$ where $x_n^{(\varepsilon)} = x_n$ for $\varepsilon = 0$, and y_n for $\varepsilon = 1$, and $k_i - k_{i-1} = 0$ or $1(k_0 = 0)$.

These equations are well defined for sequences satisfying $x_n, y_n \xrightarrow[n \rightarrow -\infty]{} a$ faster than any $\frac{1}{n^k}$, and they preserve this boundary condition. If (x_n, y_n) satisfies the same conditions for $n \rightarrow +\infty$, and if all coefficients of λ^k in $A_{N,N'} + aB_{N,N'} - D_{N,N'}$ and $B_{N,N'} - \lambda C_{N,N'}$, tend to zero faster than any $\frac{1}{N^\ell} \frac{1}{N'^{\ell'}}$ when $N, N' \rightarrow +\infty$, where $M_{N,N'} = \begin{pmatrix} A_{NN'} & C_{NN'} \\ B_{N,N'} & D_{N,N'} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \lambda x_{-N} & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & y_{N'} \\ 0 & 1 \end{pmatrix}$, then the system of equations can be rewritten with $+\infty$ replacing $-\infty$; these conditions on the lattice variables are preserved by the flow. Once again, any solution to the system, such that $x_n, y_n \rightarrow a$ and $\partial_{t_k} x_n, \partial_{t_k} y_n \rightarrow 0$, faster than any $\frac{1}{n^\ell}$ for $x \rightarrow \pm\infty$, has to satisfy the above conditions on $A_{N,N'}$ etc... Indeed, the system has to take the forms

$$\partial_k^{SG} n = \sum_s a_{k,s} \lambda^{-s} \begin{pmatrix} -\frac{a}{2} & \lambda^{-1} \\ 1 & \frac{a}{2} \end{pmatrix} n$$

$$\text{for } n = \begin{pmatrix} 1 & 1/(\lambda x_i) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -[\lambda y_{i-1} \cdots]^{-1} & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \text{ and}$$

$$\partial_k^{SG} n' = \sum_s b_{k,s} \lambda^{-s} \begin{pmatrix} \frac{a}{2} & \lambda^{-1} \\ 1 & -\frac{a}{2} \end{pmatrix} n'$$

$$\text{for } n' = \begin{pmatrix} 1 & 0 \\ 1/y_i \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 & -[\lambda x_{i-1} \cdots]^{-1} \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} t' & 0 \\ 0 & t'^{-1} \end{pmatrix}, \text{ for } |i| \text{ large enough.}$$

This implies that $(M_{N,N'})^{-1} \begin{pmatrix} -\frac{a}{2} & \lambda^{-1} \\ 1 & \frac{a}{2} \end{pmatrix} M_{N,N'} = \begin{pmatrix} \mathcal{A} & \mathcal{C} \\ \mathcal{B} & \mathcal{D} \end{pmatrix}$ should satisfy $2\mathcal{A} + a\lambda\mathcal{C} \rightarrow 0$, and $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{A} & \mathcal{C} \\ \mathcal{B} & \mathcal{D} \end{pmatrix} \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathcal{A}' & \mathcal{C}' \\ \mathcal{B}' & \mathcal{D}' \end{pmatrix}$ should satisfy $2\mathcal{A}' = a\mathcal{B}'$, or $2\mathcal{A} + a\mathcal{B} \rightarrow 0$. So we have $M_{N,N'}^{-1} \begin{pmatrix} -\frac{a}{2} & \lambda^{-1} \\ 1 & \frac{a}{2} \end{pmatrix} M_{N,N'} \rightarrow \begin{pmatrix} -\frac{a}{2} & \lambda^{-1} \\ 1 & \frac{a}{2} \end{pmatrix}$ faster than any inverse polynomial ; the conditions (2) are satisfied.

3. Finite dimensional orbits of lattice flows

Let G be the completion of $S\ell_2(\mathbf{C}((\lambda)))$, equal to the product $\pi^{-1}(B_+) \times \bar{\pi}^{-1}(N_-)$, $\pi : S\ell_2(\mathbf{C}[[\lambda]]) \rightarrow S\ell_2(\mathbf{C})$, $\lambda \mapsto 0$ and $\bar{\pi} : S\ell_2(\mathbf{C}[[\lambda^{-1}]]) \rightarrow S\ell_2(\mathbf{C})$, $\lambda^{-1} \mapsto 0$, and let A_+ be the subgroup of $\pi^{-1}(B_+)$ corresponding to the Lie algebra $a_+ = \bigoplus_{k \geq 0} \mathbf{C}\lambda^k(e + \lambda f)$, and H_- the subgroup of $\bar{\pi}^{-1}(N_-)$, corresponding to $h_- = \bigoplus_{k \geq 0} \mathbf{C}h\lambda^{-k}$.

There is a bijective correspondence between elements of $A_+ \setminus G/H_-$ and systems of variables $x_i, y_i, M_i^\pm, i \in \mathbf{Z}$, x_i, y_i scalars and M_i^\pm elements of Lie $\pi^{-1}(B_+)$, conjugate to $\lambda^{-1}e + f$ and such that $M_i^+ = \text{Ad} \begin{pmatrix} 1 & y_i \\ 0 & 1 \end{pmatrix} M_i^-, M_{i+1}^+ = \text{Ad} \begin{pmatrix} 1 & 0 \\ \lambda x_{i+1} & 1 \end{pmatrix} M_i^-$.

To see it, we express $A_+ g H_-$ as $A_+ n_+ n_- H_-$ and write $n_- = \begin{pmatrix} 1 & 1/\lambda x_0 + \dots \\ 0 & 1 \end{pmatrix}$.

$\cdot \begin{pmatrix} 1 & 0 \\ -[\lambda y_{-1} + \dots]^{-1} & 1 \end{pmatrix}$, and $A_+ g w^\pm(k) H_- = A_+ n_+^\pm(k) n_-^\pm(k) H_-$, and then pose $M_k^\pm = \text{Ad} n_+^\pm(k)^{-1} (\lambda^{-1}e + f)$ [where $w^+(k) = (w_0 w_1)^k$, $w^-(k) = (w_0 w_1)^k w_0$]. It is clear what the images of the natural flows on $A \setminus G/H_-$ are by this correspondence. Finite dimensional orbits of these flows are also in correspondence ; they are also the orbits where a linear dependance condition between some of these flows is satisfied.

Solutions (x_n, y_n) to (1), (2), satisfying the conditions $x_n(t), y_n(t) \xrightarrow[n \rightarrow \infty]{} a$ faster than any $\frac{1}{n^k}$, for any fixed t , yield solutions to the above described system on (x_i, y_i, M_i^\pm) : it suffices to identify M_i^+ and M_i^- with the matrices defined in (3) and (4).

We will now describe the finite dimensional orbits of $A_+ \setminus G/H_-$, and describe those satisfying conditions at infinity (the soliton-like orbits).

Let $g \in G$. It will lie in a finite dimensional orbit, iff a condition of the form $(\Lambda^{-1} + \sum_{k \geq 0} a_k \Lambda^{2k+1})g = g \sum_{k \leq N} b_k h \lambda^k$ ($b_N \neq 0$) is satisfied. Writing $g = n_+ b_-$, this means the existence of some $x \in \text{sl}_2(\mathbf{C}[\lambda, \lambda^{-1}])$, such that $(\Lambda^{-1} + \sum_{k \geq 0} a_k \Lambda^{2k+1})n_+ = n_+ x$, and $x b_- = b_- \sum_{k \leq N} b_k h \lambda^k$; x has to be of the form $\Lambda^{-1} +$ elements of homogeneous degree ≥ 0 and $< 2N + b_N h \lambda^N$. The vector fields (1), (2) induce on x the flows

$$(5) \quad \partial_n^{SG} x = \left[x, \left(\frac{\lambda^{-n}}{1 + \sum_{k \geq 0} a_k \lambda^{k+1}} x \right)_+ \right] \quad \text{and} \quad \partial_n^{KdV} x = \left[\left(\frac{x \lambda^n}{b_N \lambda^N + \sum_{k < N} b_k \lambda^k} \right)_-, x \right],$$

$n \geq 1$, keeping a_k and b_k fixed.

Let us indicate how to obtain (x_i, y_i, M_i^\pm) if we know $x(t)$. Suppose that $g(t)$ corresponds to the dot x_i (resp. y_i). Then $\text{Ad} \begin{pmatrix} 1 & -y_{i-1} \\ 0 & 1 \end{pmatrix} x(t)$, resp. $\text{Ad} \begin{pmatrix} 1 & 0 \\ -\lambda x_{i-1} & 1 \end{pmatrix} x(t)$ gives the matrix $x(t)$ corresponding to the previous dot, y_{i-1} and x_{i-1} respectively. That these matrices again take the form allowed for matrices $x(t)$ determines y_{i-1} and x_{i-1} : posing $x(t) = \pm b_N h \lambda^N + a_{N-1} e \lambda^{N-1} + c_N f \lambda^N +$ lower degree terms at dot x_i (resp. y_i), we deduce

$$y_{i-1}(t) = -\frac{2b_N}{c_N}(t), \quad \text{resp. } x_{i-1}(t) = -\frac{2b_N}{a_{N-1}}(t).$$

The values of the other variables $x_k(t), y_k(t)$ can be defined inductively. M_i^\pm can now be obtained as follows : $M_i^\pm = n_+^{-1} \Lambda^{-1} n_+$, so $M_i^\pm(t) = \frac{1}{1 + \sum_{k \geq 0} a_k \lambda^{k+1}} x(t)$, if $g(t)$ corresponds to the dot x_i (resp. y_i).

In general, the system (5) can be solved noticing that $x(t)$ follows an isospectral evolution, and that it corresponds to linear flows on the Jacobian of the curve $\det(\mu - x(\lambda)) = \mu^2 - \left(\sum_{k \leq N} b_k \lambda^k\right)^2 = \mu^2 - \lambda^{-1}(1 + \sum_{k \geq 0} a_k \lambda^{k+1})^2 = 0$. If this curve is smooth, its Jacobian is compact and the solutions (x_i, y_i, M_i^\pm) are expected to have a quasiperiodic behaviour. In any case, we will see that such solutions cannot have a soliton-like behaviour; such solutions will correspond to rational curves with double points.

4. Soliton solutions of the lattice flow

4.1 Integration of the system

Let us determine now the $(x(t), a_k, b_k)$ corresponding to a constant solution $x_i = y_i = a$. We have for such a solution, $N = 0$ so at dot x_i , $x = \Lambda^{-1} + b_1 h$ and $x_i(t) = -2b_1$, and at dot y_i , $x = \Lambda^{-1} - b_1 h$ and $y_i(t) = -2b_1$; so $b_1 = -\frac{a}{2}$; the constant solution corresponds to the rational curve $\mu^2 = (\frac{a}{2})^2 + \lambda^{-1}$.

Let us suppose that (x_i, y_i, M_i^\pm) has soliton behaviour, and corresponds to a finite dimensional orbit. Then by (3), (4), we have for any fixed $t = (t_k)$, $M_i^\pm(t) \xrightarrow[i \rightarrow -\infty]{} (1 + \frac{a^2 \lambda}{4})^{-1/2} \begin{pmatrix} \pm \frac{a}{2} & \lambda^{-1} \\ 1 & \mp \frac{a}{2} \end{pmatrix}$ (this means that all coordinates of $M_i^\pm(t)$ in the natural coordinate system of $S\ell_2(\mathbf{C}((\lambda)))$ has the limit indicated). This implies that $\frac{1}{1 + \sum_{k \geq 0} a_k \lambda^{k+1}} \cdot x_{(x_i), (y_i)}(t)$ have the same limit ; $x_{(x_i)}(t)$ and $x_{(y_i)}(t)$ denote the functions $x(t)$ relative to dots x_i and y_i . We have now $x_{(x_i)}(t) \rightarrow \frac{1 + \sum_{k=0}^{\infty} a_k \lambda^{k+1}}{(1 + (\frac{a}{2})^2 \lambda)^{1/2}} \begin{pmatrix} \frac{a}{2} & \lambda^{-1} \\ 1 & -\frac{a}{2} \end{pmatrix}$, and since $x_{(x_i)}(t)$ has no terms of homogeneous degree $> N$, we deduce that $\frac{1 + \sum_{k=0}^{\infty} a_k \lambda^{k+1}}{(1 + (\frac{a}{2})^2 \lambda)^{1/2}}$ is a polynomial of degree N , $\prod_{i=1}^N (1 + \gamma_i \lambda)$. So the spectral curve has equation

$$(6) \quad \mu^2 = \left(\lambda^{-1} + \left(\frac{a}{2}\right)^2\right) \prod_{i=1}^N (1 + \gamma_i \lambda)^2;$$

it is a rational curve. This phenomenon is also observed in [NMPZ], II, 10.1. Positing $(\lambda^{-1} + (\frac{a}{2})^2) \prod_{i=1}^N (1 + \gamma_i \lambda^2) = \lambda^{-1} + \sum_{i=0}^{2N} \alpha_i \lambda^i$, we get the linear dependance condition $\partial_1^{SG} + \sum_{i=1}^{2N} \alpha_i \partial_i^{KdV} = 0$ on our solution. If we assume all α_i 's to be real, the checking of $x_i(t) \rightarrow a, y_i(t) \rightarrow a$ for $t_i \rightarrow \pm\infty$, other t_i 's fixed will be reduced to a problem of a finite number of flows. (Some solutions with complex α_i 's have the same property, e.g. 1-soliton with $\text{Arg } \gamma_\varepsilon$ irrational.) From now on we suppose all α_i 's real.

Let us solve (5) in this case. Let $x(\lambda) = \begin{pmatrix} a(\lambda) & c(\lambda) \\ b(\lambda) & -a(\lambda) \end{pmatrix}$, and $b(\lambda) = \prod_{i=1}^N (1 - \frac{\lambda}{\lambda_i})$; then $a(\lambda_i)^2 = \mu(\lambda_i)^2$, so $a(\lambda_i) = \mu_i$, (λ_i, μ_i) being some point of (6), lying over λ_i . We deduce

$$(7) \quad a(\lambda) = \varepsilon \frac{a}{2} \prod_{i=1}^N \gamma_i \prod_{i=1}^N (\lambda - \lambda_i) + \sum_{i=1}^N \mu_i \prod_{\substack{j=1 \\ j \neq i}}^N \frac{\lambda - \lambda_j}{\lambda_i - \lambda_j}, \varepsilon = \pm 1.$$

Let us consider instead of the flows (5) their linear combinations

$$\partial_{t'_k} x = [(\lambda^{-k} x)_+, x], k \geq 0.$$

Note that all $\partial_{t'_n} x$ are 0 for $n < N$, and for $n \geq N$ they will all be proportional to $[x_-, x]$. The vector fields $\partial_{t'_k}$ induce on $b(\lambda)$ the flows $\frac{1}{2} \partial_{t'_k} b = (\lambda^{-k} b)_+ a - (\lambda^{-k} a)_+ b$ (here $(\lambda^i)_+ = \lambda^i 1_{i>0}$). Evaluation at λ_i gives $\frac{1}{2} \prod_{j \neq i} (1 - \frac{\lambda_i}{\lambda_j}) \frac{\partial_{t'_k} \lambda_i}{\lambda_i} = \mu_i (\lambda^{-k} b)_+ (\lambda_i) = \mu_i \lambda_i \cdot$ coefficient of λ^k in $\frac{b(\lambda)}{\lambda - \lambda_i} = \mu_i (-1)^{k+1} \sum_{\substack{i_\alpha \neq i \\ i_\alpha \text{ all different}}}^{k+1} \frac{1}{\lambda_{i_1} \dots \lambda_{i_k}}$, so

$$\partial_{t'_k} \lambda_i = 2 \frac{(-1)^{k+1} \mu_i \lambda_i}{\prod_{j \neq i} (\lambda_j - \lambda_i)} \sum_{\substack{j_\alpha \neq i \\ \text{all different}}} \lambda_{j_1} \dots \lambda_{j_{N-k-1}}.$$

Consider now the forms $\omega_s = \lambda^s \frac{d\lambda}{\mu}$, $s = -1, \dots, N-2$, and the functions $F_s(t') = \sum_{i=1}^N \int_{P_0}^{(\lambda_i(\tau), \mu_i(\tau))} \omega_s$ (P_0 a fixed point on the curve). They obey

$$\partial_{t'_k} F_s = 2(-1)^{k+1} \sum_{i=1}^N \frac{\lambda_i^{s+1}}{\prod_{j \neq i} (\lambda_j - \lambda_i)} \sum_{\substack{j_\alpha \neq i \\ \text{all different}}} \lambda_{j_1} \dots \lambda_{j_{N-k-1}} = -2\delta_{k,s+1}.$$

by a Vandermonde system computation. So $F_s(t') = -2t'_{s+1} + cst$. But

$$F_s(t') = \sum_{i=1}^N \int_{\lambda_0}^{\lambda_i} \frac{\lambda^s d\lambda}{((\frac{a}{2})^2 + \lambda^{-1})^{1/2} \prod_{i=1}^N (1 + \gamma_i \lambda)} = \sum_{i=1}^N \int_{\tau_0}^{\tau_i} \frac{-2(\tau^2 - (\frac{a}{2})^2)^{N-s-2}}{\prod_{j=1}^N (\tau^2 + \gamma_j - (\frac{a}{2})^2)} d\tau,$$

with $\tau^2 = (\frac{a}{2})^2 + \lambda^{-1}$; more precisely $\tau = \mu / \prod_{i=1}^N (1 + \gamma_i \lambda)$.

Assuming all γ_i 's different, the fraction in τ is decomposed as

$$-\sum_{i=1}^N \frac{(-\gamma_i)^{N-s-2}}{\prod_{j \neq i} (\gamma_j - \gamma_i) \sqrt{(\frac{a}{2})^2 - \gamma_i}} \left(\frac{1}{\tau - \sqrt{(\frac{a}{2})^2 - \gamma_i}} - \frac{1}{\tau + \sqrt{(\frac{a}{2})^2 - \gamma_i}} \right),$$

for $i = -1, \dots, N-2$. This gives

$$(8) \quad \sum_{i,j=1}^N \frac{(-\gamma_i)^{N-s-2}}{2\sqrt{(\frac{a}{2})^2 - \gamma_i} \prod_{k \neq i} (\gamma_k - \gamma_i)} \ln \frac{\tau_j - \sqrt{(\frac{a}{2})^2 - \gamma_i}}{\tau_j + \sqrt{(\frac{a}{2})^2 - \gamma_i}} = t'_{s+1} + C_{s+1}$$

C_{s+1} are constants.

If we pose $C(\lambda) = \lambda^{-1} \prod_{i=1}^N (1 - \frac{\lambda}{\nu_i})$ and $\sigma_i = \mu(\nu_i)/\prod_{j=1}^N (1 + \gamma_j \nu_i)$, we have similarly

$$(8bis) \quad \sum_{i,j=1}^N \frac{(-\gamma_i)^{N-s-2}}{2\sqrt{(\frac{a}{2})^2 - \gamma_i} \prod_{k \neq i} (\gamma_k - \gamma_i)} \ln \frac{\tau_j - \sqrt{(\frac{a}{2})^2 - \gamma_i}}{\tau_j + \sqrt{(\frac{a}{2})^2 - \gamma_i}} = t'_{s+1} + C'_{s+1}$$

4.2. One-soliton solutions

Consider the case $N = 1$. $\gamma_1 = \gamma$ is real. We find, with $\varepsilon = -1$

$$(9) \quad \tau_1 = \sqrt{(\frac{a}{2})^2 - \gamma} \frac{1 + K e^{2t'_0} \sqrt{(\frac{a}{2})^2 - \gamma}}{1 - K e^{2t'_0} \sqrt{(\frac{a}{2})^2 - \gamma}}, \quad \sigma_1 = \sqrt{(\frac{a}{2})^2 - \gamma} \frac{1 + K' e^{2t'_0} \sqrt{(\frac{a}{2})^2 - \gamma}}{1 - K' e^{2t'_0} \sqrt{(\frac{a}{2})^2 - \gamma}}$$

K, K' are (complex) constants of integration, related by

$$\frac{K'}{K} = \frac{\frac{a}{2} - \sqrt{(\frac{a}{2})^2 - \gamma}}{\frac{a}{2} + \sqrt{(\frac{a}{2})^2 - \gamma}}.$$

For the solution to have the properties $\lambda_1(t'_0) \rightarrow -\frac{1}{\gamma}$, $\mu_1(t'_0) \rightarrow -\frac{1}{\gamma}$ for $t'_0 \rightarrow \pm\infty$, we need $\sqrt{(\frac{a}{2})^2 - \gamma}$ to be real ; so the condition is $\gamma < (\frac{a}{2})^2$. We note that the values of τ_1, σ_1 are independant of the sign of $\sqrt{(\frac{a}{2})^2 - \gamma}$. We compute

$$(10) \quad x_{i-1}(t'_0) = a \frac{(1 - K e^{t''_0})^2}{(1 - K' e^{t''_0})(1 - \frac{K^2}{K'} e^{t''_0})}, \quad y_{i-1}(t'_0) = a \frac{(1 - K' e^{t''_0})^2}{(1 - K e^{t''_0})(1 - \frac{K'^2}{K} e^{t''_0})}$$

with $t''_0 = 2\sqrt{(\frac{a}{2})^2 - \gamma}t'_0$. We have $x_{i-1}(t'_0), y_{i-1}(t'_0) \rightarrow a$ for $t'_0 \rightarrow \pm\infty$.

Replacing ε by 1 in (7) amounts to replacing a by $-a$. (10) can be considered as a one-soliton solution to the system.

The values of $x_i(t'_0), y_i(t'_0)$ are given by formulae (10), with K and K' multiplied by $(\frac{K'}{K})^2$: it means that the lattice translation of the solution corresponds to a time translation of

$$(11) \quad \Delta t'_0 = \frac{1}{\sqrt{(\frac{a}{2})^2 - \gamma}} \ln \left(\frac{\frac{a}{2} - \sqrt{(\frac{a}{2})^2 - \gamma}}{\frac{a}{2} + \sqrt{(\frac{a}{2})^2 - \gamma}} \right)^2$$

Note that up to time shift, complex solitons depend on one variable ($\text{Arg } K$) and real solitons on no variable.

4.2 Breather solutions.

Let us consider now the case $N = 2$, $\gamma_2 = \bar{\gamma}_1$ not real. Rather than solve (7) explicitly, let us make the following qualitative remarks. Complex solutions depend (up to real shifts in t'_0 and t'_1) on two real variables ($\text{Im}C_0$ up to integral multiples of $\text{Re}\frac{2\pi\gamma_1}{2\sqrt{(\frac{a}{2})^2-\gamma_1}(\gamma_1-\gamma_2)}$ and $\text{Im}C_1$ up to integral multiples of $\text{Re}\frac{2\pi}{2\sqrt{(\frac{a}{2})^2-\gamma_1}(\gamma_1-\gamma_2)}$). The conditions for the solution to be real are $\tau_2 = \bar{\tau}_1$, or τ_1 and τ_2 real, so the $T_i = \sum_{j=1}^2 \ln \frac{\tau_j - \sqrt{(\frac{a}{2})^2 - \gamma_i}}{\tau_j + \sqrt{(\frac{a}{2})^2 - \gamma_i}}$ should be conjugate for $i = 1, 2$ and this means that C_0 and C_1 should be real : again the real solutions depend on no additional variable (up to time shifts). Let us again consider a complex solution and let us study its evolution for large times with $\alpha t'_0 + \beta t'_1$ fixed (α, β real not both zero). Since $T_1 = 2\sqrt{(\frac{a}{2})^2 - \gamma_1}(t'_0 + \gamma_2 t'_1)$, $T_2 = 2\sqrt{(\frac{a}{2})^2 - \gamma_2}(t'_0 + \gamma_1 t'_2)$, $\text{Re } T_1$ and $\text{Re } T_2$ tend to infinity except if $(1 - \gamma_2 \frac{\alpha}{\beta})/\sqrt{(\frac{a}{2})^2 - \gamma_2}$ is pure imaginary (this fixes one value of $\frac{\alpha}{\beta}$). Using

$$(12) \quad \tau_1 \tau_2 (1 - e^{T_i}) - \sqrt{(\frac{a}{2})^2 - \gamma_i} (1 + e^{T_i}) (\tau_1 + \tau_2) + ((\frac{a}{2})^2 - \gamma_i) (1 - e^{T_i}) = 0, i = 1, 2,$$

we see that τ_1 and τ_2 tend to $\pm\sqrt{(\frac{a}{2})^2 - \gamma_{1,2}}$ so x_i and y_i tend to a . Let us study now the behaviour of these solutions w.r.t. lattice periodicity. The lattice translation corresponds to some (*a priori* complex) shift in times $(t'_0, t'_1) \mapsto (t'_0 + \Delta_0, t'_1 + \Delta_1)$. Δ_0 and Δ_1 should be analytic functions in γ_1 and γ_2 ; in particular, let us compute them in the case where γ_1 and γ_2 are real ($\gamma_1 \neq \gamma_2$). In a region

$$|t'_0 + \gamma_2 t'_1| \ll |t'_0 + \gamma_1 t'_1|,$$

the solution $x_i(t'_0, t'_1)$ tends to $x_i^{(\gamma_1)}(t'_0 + \gamma_2 t'_1)$, where $x_i^{(\gamma_1)}(t'_0)$ is the soliton solution with parameter γ_1 described before. In this situation, the lattice translation corresponds to the shift (11), so $\Delta_0 + \gamma_2 \Delta_1 = \frac{2}{\sqrt{(\frac{a}{2})^2 - \gamma_1}} \ln \frac{\frac{a}{2} - \sqrt{(\frac{a}{2})^2 - \gamma_1}}{\frac{a}{2} + \sqrt{(\frac{a}{2})^2 - \gamma_2}}$; we have the same equality exchanging γ_1 and γ_2 , so

$$\Delta_0 = \frac{1}{\gamma_1 - \gamma_2} \sum_{i=1}^2 \frac{2\gamma_i (-1)^i}{\sqrt{(\frac{a}{2})^2 - \gamma_i}} \ln \frac{\frac{a}{2} - \sqrt{(\frac{a}{2})^2 - \gamma_i}}{\frac{a}{2} + \sqrt{(\frac{a}{2})^2 - \gamma_i}},$$

$$\Delta_1 = \frac{1}{\gamma_1 - \gamma_2} \sum_{i=1}^2 \frac{2(-1)^i}{\sqrt{(\frac{a}{2})^2 - \gamma_i}} \ln \frac{\frac{a}{2} - \sqrt{(\frac{a}{2})^2 - \gamma_i}}{\frac{a}{2} + \sqrt{(\frac{a}{2})^2 - \gamma_i}}.$$

We see that in the case $\gamma_2 = \bar{\gamma}_1$, Δ_0 and Δ_1 are real. Note that the times direction (Δ_0, Δ_1) corresponds to a direction where the solution tends to a ; we have then $x_{i+N}(t'_0, t'_1) = x_i(t'_0 + N\Delta_0, t'_1 + N\Delta_1) \rightarrow a$ when $N \rightarrow \infty$. One can easily show that this convergence is exponential in N . So the solutions found here (with $\gamma_2 = \bar{\gamma}_1$) have soliton-like behaviour (in the sense of 4.1) and can be thought of as analogues of the breathers (despite the fact that solitons are not charged).

4.4 N -soliton solutions.

Let us turn now to the general case : we have p real poles $\gamma_1, \dots, \gamma_p$, ($\gamma_i < (\frac{a}{2})^2$) and q pairs of complex poles $\gamma_1, \bar{\gamma}_1, \dots, \gamma_q, \bar{\gamma}_q$; all pairwise different. We will show that the corresponding solution has soliton-like behaviour, and that it corresponds to the scattering of p solitons and q breathers. Now $N = p + 2q$. Let us discuss the effect of a large translation in times along some vector $(\alpha_0, \dots, \alpha_{N-1})$. (8) can be solved by

$$(13) \quad \sum_{j=1}^N \ln \frac{\tau_j - \sqrt{(\frac{a}{2})^2 - \gamma_i}}{\tau_j + \sqrt{(\frac{a}{2})^2 - \gamma_i}} = 2(-1)^i \sqrt{(\frac{a}{2})^2 - \gamma_i} \sum_{j=1}^N \sum_{\substack{i_{\alpha_k} \neq i \\ \text{all different}}} \gamma_{i_{\alpha_1}} \cdots \gamma_{i_{\alpha_{j-1}}} (t'_{j-1} + C_{j-1}) = T_i$$

As long as $\operatorname{Re} \left[2\sqrt{(\frac{a}{2})^2 - \gamma_i} \sum_{j=1}^N \sum_{\substack{i_{\alpha_k} \neq i \\ \text{all different}}} \gamma_{i_{\alpha_1}} \cdots \gamma_{i_{\alpha_{j-1}}} \alpha_{j-1} \right]$ are all non zero, all $\operatorname{Re}(T_j)$ will tend to infinity, which means that τ_j tend to $\sqrt{(\frac{a}{2})^2 - \gamma_j}$ (up to order) and $x_a(t'_i), y_\alpha(t'_i)$ tends to a .

Let us discuss now the effect of a lattice translation. It corresponds to some shift in times $(t'_i) \mapsto (t'_i + \Delta_i)$, $i = 0, \dots, N-1$. We again compute Δ_i by analytic continuation of the case where all γ_i 's are real. Fix some index i . In a region $|T_i| \ll |T_k|$, $k \neq i$, we have $\tau_k \rightarrow \pm \sqrt{(\frac{a}{2})^2 - \gamma_k}$ and $x_\alpha(t'_0, \dots, t'_{N-1})$ tends to $x_\alpha(T_i + \text{const})$. Proceeding as we did previously, we find

$$\Delta_\alpha = \sum_{i=1}^N \frac{2(-\gamma_i)^{N-1-\alpha}}{\sqrt{(\frac{a}{2})^2 - \gamma_i}} \ln \frac{\frac{a}{2} - \sqrt{(\frac{a}{2})^2 - \gamma_i}}{\frac{a}{2} + \sqrt{(\frac{a}{2})^2 - \gamma_i}} .$$

In our case, Δ'_α 's are real. Since $\operatorname{Re}(2(-1)^i \sqrt{(\frac{a}{2})^2 - \gamma_i} \sum_{j=1}^N \sum_{\substack{i_{\alpha_k} \neq j \\ \text{all different}}} \gamma_{i_{\alpha_1}} \cdots \gamma_{i_{\alpha_{j-1}}}) = \operatorname{Re} \left(\ln \frac{\frac{a}{2} - \sqrt{(\frac{a}{2})^2 - \gamma_i}}{\frac{a}{2} + \sqrt{(\frac{a}{2})^2 - \gamma_i}} \right)$, it is not zero, by the assumptions. We deduce again that for any (t'_i) , $x_\alpha(t'_j)$ and $y_\alpha(t'_i)$ tend to a as $\alpha \rightarrow \pm\infty$; this convergence is exponential in α , as is easily shown. So the solution has soliton behaviour.

Let us indicate the regions of large times where the solution does not tend to a : we have $|T_i| \ll |T_k|, |T_\ell + \bar{T}_\ell|$, for $i \leq p$, $k \leq p$ and $k \neq 0$, $\ell \geq p+1$; and $|T_\ell + \bar{T}_\ell| \ll |T_k|, |T_{\ell'} + \bar{T}_{\ell'}|$, for $i \leq p$, $\ell, \ell' \geq p+1$, $\ell \neq \ell'$. They correspond to solitons and breathers with parameters γ_i , $i = 1, \dots, N-2$.

Let us compute now the phase shift of the particle γ_i , due to the scattering with other particles.

We find, when τ_j goes from $\sqrt{(\frac{a}{2})^2 - \gamma_j}$ to $-\sqrt{(\frac{a}{2})^2 - \gamma_j}$, for $j \neq i$, $\Delta T_i = \sum_{j \neq i} \ln \frac{\sqrt{(\frac{a}{2})^2 - \gamma_i} - \sqrt{(\frac{a}{2})^2 - \gamma_j}}{\sqrt{(\frac{a}{2})^2 - \gamma_i} + \sqrt{(\frac{a}{2})^2 - \gamma_j}}$ if $i \leq p$, and $\Delta(T_\ell + \bar{T}_\ell) = \sum_{j \neq i, \bar{i}} 2\operatorname{Re} \ln \frac{\sqrt{(\frac{a}{2})^2 - \gamma_i} - \sqrt{(\frac{a}{2})^2 - \gamma_j}}{\sqrt{(\frac{a}{2})^2 - \gamma_i} + \sqrt{(\frac{a}{2})^2 - \gamma_j}}$ if $i > p$. So the phase shifts are the sums of two-particle contributions ; this is a situation of elastic scattering.

4.5. Multiple-poles solutions.

Assume that the poles $\gamma_0, \gamma_j, \bar{\gamma}_j (i \leq p, j \geq p+1)$ have multiplicities α_i, α_j . Formula (8) is then replaced by

$$(14) \quad \sum_{i=1}^{p+q} \frac{1}{\prod_{k \neq i} (\gamma_k - \gamma_i)^{\alpha_i + \alpha_k}} \frac{1}{(\alpha_i - 1)!} \left(\frac{\partial}{\partial \gamma_i} \right)^{\alpha_i - 1} \left[\frac{(-\gamma_i)^{N-s-2}}{2\sqrt{(\frac{a}{2})^2 - \gamma_i}} \sum_{j=1}^N \ln \frac{\tau_j - \sqrt{(\frac{a}{2})^2 - \gamma_i}}{\tau_j + \sqrt{(\frac{a}{2})^2 - \gamma_i}} \right]$$

$$= t'_{s+1} + cst, \text{ where } N = \sum_{i=1}^{p+q} \alpha_i, \text{ because } \lim_{\gamma_i \rightarrow \gamma} \sum_{i=1}^k \frac{F(\gamma_i)}{\prod_{j \neq i} (\gamma_i - \gamma_j)} = \frac{F^{(k)}(\gamma)}{k!},$$

which is solved by

$$(15) \quad \left(\frac{\partial}{\partial \gamma_i} \right)^\beta \sum_{j=1}^N \ln \frac{\tau_j - \sqrt{(\frac{a}{2})^2 - \gamma_i}}{\tau_j + \sqrt{(\frac{a}{2})^2 - \gamma_i}} = T_{i,\beta}, \quad \beta = 0, \dots, \alpha_i - 1,$$

where the $T_{i,\beta}$ are linear combinations of the times t'_{s+1} . The effect of a lattice translation and the large time behaviour of this system are studied in the same way as for the simple poles solutions : so multiple poles again define solitonic solutions, whose scattering can be computed similarly.

We note two main differences with the continuous case : absence of topological charge, and no fermionic nature of the solitons.

5. Periodic solutions of the lattice flow.

We now examine the possibility of finite dimensional orbits of the lattice flow yielding periodic (in i) solutions $x_i(t), y_i(t)$. Let ν be this period. The condition for periodicity is the existence of an element b in \widehat{N}_+ , of the form $\begin{pmatrix} * \lambda^{\nu-1} + \dots * \lambda^{\nu-1} + \dots \\ * \lambda^\nu + \dots *' \lambda^\nu + \dots \end{pmatrix}$, $*' \neq 0$ (for example), commuting to $x_{(x_i)}(t)$. This condition is preserved by the flows : b has to follow the equations $\partial_{t_n}^{SG} b = [b, (\frac{\lambda^{-n}}{1 + \sum a_k \lambda^{k+1}} x)_+] = [(\frac{\lambda^{-n}}{1 + \sum a_k \lambda^{k+1}} x)_-, b]$ and $\partial_{t_n} b = [(\frac{x \lambda^n}{b_N \lambda^N + \sum_{k < N} b_k \lambda^k})_-, b] = -[(\frac{x \lambda^n}{b_N \lambda^N + \sum_{k < N} b_k \lambda^k})_+, b]$ which clearly preserve its form. Writing $b = P + Qx$, P, Q rational fractions in λ , the conditions on P and Q are : P polynomial, Qx polynomial and $P^2 + Q^2 \det x = 1$. Thus the condition is the existence of a meromorphic function on the spectral curve, with only zeroes and poles at the points at infinity ∞^\pm . The order of this zero (or pole) is ν , and we see that we have $\nu(\infty^+ - \infty^-) = 0$ is the Jacobian of the curve; ν is also the lattice period. Thus the periodicity condition is that the difference of the infinity points is torsion is the Jacobian of the spectral curve.

We compute now the dimension of the space of periodic solutions among all spectral curves of degree N . Writing $\det x = -\lambda^{-1} \prod_{i=1}^{2N+1} (1 - \frac{\lambda}{\lambda_i})$ and $Q = \frac{R}{S}, R, S$ polynomials of minimal degree, we find that λ should divide R . We find also that S should divide $\prod_{i=1}^{2N+1} (1 - \frac{\lambda}{\lambda_i})$, and also x ; but we can restrict ourselves without loss of generality, to the case where no factor of $\prod_{i=1}^{2N+1} (1 - \frac{\lambda}{\lambda_i})$ divides x . Let now $R = \lambda R_0, S = 1$; the condition Qx polynomial is now satisfied, and we need only to

satisfy $P^2 = 1 + R_0^2 \lambda \prod_{i=1}^{2N+1} (1 - \frac{\lambda}{\lambda_i})$. The degrees of P and R_0 are ν and $\nu - (N+1)$; let us write $\frac{P}{R_0} = \left[\lambda \prod_{i=1}^{2N+1} (1 - \frac{\lambda}{\lambda_i}) \right]^{1/2} \left[1 + \frac{K}{\lambda^{2\nu-2(N+1)}} + O(\lambda^{-2\nu-2(N+1)}) \right]$; we obtain $\frac{P_0}{R_0} = ([\lambda \prod_{i=1}^{2N+1} (1 - \frac{\lambda}{\lambda_i})]^{1/2})_- + \frac{k'}{\lambda^{2\nu-(N+1)}}$, for P_0 the remainder of the division of P by R_0 and $K' = K(-\prod_{i=1}^{2N+1} \lambda_i)^{-1/2}$. The space of possible fractions has dimension $2(\nu - N - 1)$, and $2\nu - (N + 2)$ equations must be satisfied in it; this imposes N conditions on the $(\lambda_i)_{i=1,\dots,2N+1}$. It would be interesting to see if some natural Poisson brackets on the (λ_i) can be introduced, and what kind of submanifold the periodic solutions will be in the space of all solutions (or of hyperelliptic curves) with respect to this Poisson geometry.

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Centre de mathématiques
 URA 169 du CNRS
 Ecole Polytechnique
 91128 Palaiseau Cedex
 France